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A new family based on lifetime distribution: Bivariate Weibull-G models based on Gaussian copula

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ABSTRACT

Copula method plays an essential rule to study the dependence between data variables especially in bivariate distribution. It is noted that some bivariate models are constructed with uncomplete information of distributions. Copula improves the reliability of applications such as flood peak. Weibull distribution is a popular used in engineering, theory, medical and survival analysis. Despite its spread, it is known that the Weibull distribution could not implement the data set with non-monotone failure rate. In such case, many papers have suggested a modification and generalization of Weibull model. One of generalization is made through the baseline distribution by adding more shape parameters. The main purpose of our paper is to present some new bivariate Weibull models with respect to G cumulative distributions of baseline distribution. This approach converges the power series of probability distribution. We use the copula function to construct the bivariate Weibull distribution. The proposed models provide high flexibility and can be used effectively for modeling dataset with a different structure. We provide special cases in details namely; bivariate Weibull-exponential, bivariate Weibull-Rayleigh and bivariate Weibull Chi-square. We use Gaussian copula function to merge the dependent distributions, this copula is popular used in various applications like econometrics and finance. We discuss some structural properties of the proposed models. In order to estimate the model parameters, we discuss parametric methods via maximum likelihood estimation and modified maximum likelihood methods. In addition, we use the moment methods as semi-parametric methods for parameters estimations. Finally, Simulations are studied to illustrate methods of inference discussed and study the performance of new distributions.

1. Introduction

An increased scientific effort has been made to propose generators for continuous families of univariate distributions of only one random variable through the baseline distribution by adding more shape parameters. Other generators such as, beta-G distributions presented by Eugene et al. (2002), and exponentiated generalized–G distributions proposed by Cordeiro et al. (2013), some methods are discussed in Nadarajah and Rocha (2015). The class of Weibull G distributions (WG) has received an increasing amount of attention in recent years. Many studies conducted based on the properties and inferences of Weibull G distributions with a consideration to their applications. Our aim is to propose new bivariate Weibull G distributions. The class of bivariate models proposed, in this paper, presents more flexible in WG marginals and in dependence structure. Bivariate Weibull-exponential, Bivariate Weibull-Rayleigh and Bivariate Weibull Chi-square are considered with several fields of application. The (WG) probability density function (PDF) is the following:

$$f(t,\alpha,\beta,\delta) = \frac{\alpha}{\beta^{\alpha}}\frac{g(t,\delta)}{1-G(t,\delta)}\left\{-\frac{log[1-G(t,\delta)]}{\beta}\right\}^{\alpha-1} exp\left\{-\left[-\frac{log[1-G(t,\delta)]}{\beta}\right]^{\alpha}\right\},\ t\geq 0 \quad (1)$$

The $g(t,\delta)$ is the probability density distribution PDF and $G(t,\delta)$ is a cumulative distribution CDF of t, where t in the range of $g(t,\delta)$. The β is known as scale parameter and takes positive values, while α

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considers as the shape parameter with positive values. The Weibull G distribution function (Cdf) is as follows:

$$F(t,\alpha,\beta,\delta) = 1 - exp\left\{\left[-\frac{log[1-G(t,\delta)]}{\beta}\right]^{\alpha}\right\} \quad (2)$$

The Weibull G quantile function is

$$F^{-1}(P,\delta) = G^{-1}\left(1 - exp\left\{-\beta[-log(1-P)]^{1/\alpha}\right\}\right) \quad (3)$$

where 0≤ p ≤1, the scale parameter β takes positive values, while α considers as the shape parameter with positive values. The Weibull G hazard rate functions is given by

$$h(t,\alpha,\beta,\delta) = \frac{\alpha}{\beta^{\alpha}}\frac{g(t,\delta)}{1-G(t,\delta)}\left\{-\frac{log[1-G(t,\delta)]}{\beta}\right\}^{\alpha-1} exp\left\{-2\left[-\frac{log[1-G(t,\delta)]}{\beta}\right]^{\alpha}\right\}. \quad (4)$$

Note that pdf, cdf, quantile and random numbers of the Weibull G (WG) distribution are discussed by Alzaatreh et al. (2013a). Class Weibull G distributions discussed in the several papers such that, the Weibull Pareto distribution Alzaatreh et al. (2013a), Weibull G distributions also due to Alzaatreh et al. (2013b), Nadarajah et al., (2015), and Nadarajah and Rocha (2015). Copulas are a general tool to construct multivariate distributions and study the dependence structure between random variables. We show that copulas can used to solve many important problems. Several bivariate and multivariate lifetime distributions are suggested using several methods of constructing bivariate and multivariate distributions and copula functions have been proposed by Nelsen (1999), Trivedi and Zimmer (2007), Adham et al. (2016), Abd Elaal (2017), and Abd Elaal and Alzahrani (2017), among others. The purpose of this article is to explain bivariate Weibull G(BWG) models incorporate with Gaussian copula function. The organization of paper as follows. Some special models of univariate WG distributions are explained in next Section. New class of bivariate Weibull G (BWG) models which uses Gaussian copula as a merge index is introduced in Section 3. Section 4 illustrates some special models of bivariate Weibull G distributions based on Gaussian copula. Parametric and semi-parametric methods are used to estimate the parameters of BWG models are presented in Section 5. In Section 6, goodness of fit test statistics for the bivariate Weibull G (BWG) models is used to check the performance of the selected copula function. Finally, the performance of the suggested models using a simulation data is explained in Section 7. Section 8 discusses the simulation results. Finally, the conclusion is drawn in section 9.

2. Special models of univariate WG distributions

This section presents some special distribution of univariate WG distributions.

2.1. Weibull-uniform distribution

Consider $g(t)$ with uniform distribution on the interval (0,r), r > 0.

$$g(t;r) = \frac{1}{r}\ ,\quad 0 < x < r < \infty,$$

and

$$G(t;r) = \frac{t}{r}.$$

The Weibull-Uniform (WU) distribution PDF and CDF are formed as, respectively

$$f(t,\alpha,\beta,r) = \frac{\frac{\alpha}{r}}{\beta^{\alpha}\left[1-\frac{t}{r}\right]}\left\{-\frac{log\left[1-\frac{t}{r}\right]}{\beta}\right\}^{\alpha-1} exp\left\{-\left[-\frac{log\left[1-\frac{t}{r}\right]}{\beta}\right]^{\alpha}\right\} \quad (5)$$

and

$$F(t,\alpha,\beta,r) = 1 - exp\left\{-\left[-\frac{log\left[1-\frac{t}{r}\right]}{\beta}\right]^{\alpha}\right\} \quad (6)$$

where , $0 < t < r < \infty$, *and* $\alpha,\beta,r > 0$.

2.2. Weibull-exponential distribution

Now, if $g(t)$ is an Exponential distribution. The PDF and CDF are

$$g(t;r) = r\exp(-rt),\quad 0 < t < r < \infty,$$

and

$$G(t;r) = 1 - \exp(-rt),$$

and then the Weibull-Exponential (WEXP) distribution PDF and CDF are formed as, respectively for $t,\alpha,\beta,r > 0$

$$f(t,\alpha,\beta,r) = \frac{\alpha r^{\alpha}}{\beta^{2\alpha-1}}t^{\alpha-1}exp\left\{-\left[\frac{rt}{\beta}\right]^{\alpha}\right\} \quad (7)$$

and

$$F(t,\alpha,\beta,r) = 1 - exp\left\{-\left[\frac{rt}{\beta}\right]^{\alpha}\right\} \quad (8)$$

2.3. Weibull-Rayleigh distribution

Let $g(t)$ is Rayleigh distribution. We have

$$g(t;r) = 2rt\exp(-rt^2),,\ 0 < t < r < \infty,$$

and

$$G(t;r) = 1 - \exp(-rt^2),$$

and then the Weibull-Rayleigh (WR) distribution PDF and CDF are formed as, respectively

$$f(t,\alpha,\beta,r) = \frac{2\alpha r^{\alpha}}{\beta^{2\alpha-1}}t^{2\alpha-1}exp\left\{-\left[\frac{rt^2}{\beta}\right]^{\alpha}\right\} \quad (9)$$

$$F(t,\alpha,\beta,r) = exp\left\{-\left[\frac{rt^2}{\beta}\right]^{\alpha}\right\},\ for\ t,\alpha,\beta,r > 0. \qquad (10)$$

2.4. Weibull-Chi-square distribution

Let $g(t)$ is Chi-square distribution. We have

$$g(t;r) = \frac{2^{-\frac{r}{2}}}{\Gamma(\frac{r}{2})}\ t^{\frac{r}{2}-1}\exp\left(-\frac{t}{2}\right);$$

where $t,\alpha,\beta,r > 0$

and

$$G(t;r) = \frac{\Gamma(t,\frac{r}{2})}{\Gamma(\frac{r}{2})}$$

where $\Gamma\left(t,\frac{r}{2}\right)$ is incomplete gamma and then Weibull-Chi-square (WCHI) distribution PDF and CDF distribution are formed as, respectively, for $t,\alpha,\beta,r > 0$,

$$f(t,\alpha,\beta,r) = \frac{\alpha\frac{2^{-\frac{r}{2}}}{\Gamma(\frac{r}{2})}t^{\frac{r}{2}-1}\exp\left(-\frac{t}{2}\right)}{\beta^{\alpha}\left[1-\frac{1*\Gamma(t,\frac{r}{2})}{\Gamma(\frac{r}{2})}\right]}\left\{-\frac{log\left[1-\frac{1-\Gamma(t,\frac{r}{2})}{\Gamma(\frac{r}{2})}\right]}{\beta}\right\}^{\alpha-1} exp\left\{-\left[-\frac{log\left[1-\frac{1-\Gamma(t,\frac{r}{2})}{\Gamma(\frac{r}{2})}\right]}{\beta}\right]^{\alpha}\right\}, \qquad (11)$$

and

$$F(t,\alpha,\beta,r) = 1 - exp\left\{-\left[-\frac{log\left[1-\frac{1-\Gamma(t,\frac{r}{2})}{\Gamma(\frac{r}{2})}\right]}{\beta}\right]^{\alpha}\right\};\ t,\alpha,\beta,r > 0 \qquad (12)$$

The data structure of the WG distributions can take different shapes such as bathtub, symmetrical, J shaped, skewed, various shapes of bathtub. This fact implies that the WG and BWG distributions provide more flexibility and deal with different data sets with various shapes.

3. Bivariate Weibull-G distributions based on Gaussian copula

The concept of copula suggested and derived by (Sklar, 1959), stated that any multivariate distribution can be disintegrated to a copula and its continues marginal. For the bivariate case, copulas are used to link two marginal distributions with joint distribution such that for every bivariate distribution function $F(t_1,t_2)$ with continuous marginal $F(t_1), F(t_2)$, there exist a unique copula function C as follows:

$$F(t_1,t_2) = C\{F(t_1),F(t_2)\}, \qquad (13)$$

where $(t_1,t_2) \in (-\infty,\infty)\times(-\infty,\infty)$.

The pdf function of bivariate distribution gives as

$$f(t_1,t_2) = f_1(t_1)f_2(t_2)c(F_1(t_1),F_2(t_2)) \qquad (14)$$

where $c(F_1(t_1),F_2(t_2)$ is the density function of copula see (Nelsen, 1999). Several copula functions are used to construct BWG distributions with WG marginal given by (1). In this article, we will apply Gaussian copula to construct BWG distributions. The formula of Gaussian copula is

$$C(u,v) = \varphi_{\Sigma}(\varphi^{-1}(u),\varphi^{-1}(v)) \qquad (15)$$

where φ_{Σ} denotes the distribution function of a bivariate standard normal random variable and φ^{-1} represents its inverse. Now, the joint PDF of T_1 and T_2 based on Gaussian copula becomes

$$f(t_1,t_2) = f_1(t_1)f_2(t_2)\left\{\frac{1}{\sqrt{1-\rho^2}}(\exp[\tfrac{-\rho}{2(1-\rho^2)}\{\rho(z_1^2+z_2^2) - 2z_1\ z_2\ \}])\right\}, \qquad (16)$$

Where $\rho \in [-1,1]$ is a dependence parameter and $f_1(t_1), f_2(t_2)$ is the density function of WG distributions in (1).

4. The proposed models: Special models of bivariate Weibull-G distributions based on Gaussian copula

The class of Bivariate Weibull G distributions based on Gaussian copula is presented in this section, we illustrate our proposed special models in next sub-sections.

4.1. Bivariate Weibull-uniform distribution

Suppose we have two random variables where their marginals followed Weibull-Uniform (WU) distribution. Hence, the bivariate Weibull-Uniform (BWU) distribution PDF, the joint PDF of T_1 and T_2 based on Gaussian copula, is

$$f\left(t_1,t_2,\underline{\alpha},\underline{\beta},\underline{r}\right) = \prod_{j=1}^{2}\frac{\frac{\alpha_j}{r_j}}{\beta_j^{\alpha_j}\left[1-\frac{t}{r_j}\right]}\left\{-\frac{log\left[1-\frac{t}{r_j}\right]}{\beta_j}\right\}^{\alpha_j-1} exp\left\{-\left[-\frac{log\left[1-\frac{t}{r_j}\right]}{\beta_j}\right]^{\alpha_j}\right\} \left\{\frac{1}{\sqrt{1-\rho^2}}(\exp[\tfrac{-\rho}{2(1-\rho^2)}\{\rho(z_1^2+z_2^2) - 2z_1z_2\}])\right\}, \qquad (17)$$

where

$$0 < t_j < r_j < \infty,\quad \alpha_j,\beta_j,r_j > 0,$$

and $\rho \in [-1,1]$ is a correlation parameter of model also known as a dependence parameter.

4.2. Bivariate Weibull-exponential distribution

Let the marginals have Weibull-Exponential (WEXP) distribution then the bivariate Weibull-Exponential (BWEXP) distribution PDF is

$$f\left(t_1,t_2,\underline{\alpha},\underline{\beta},\underline{r}\right) = \prod_{j=1}^{2}\frac{\alpha_j r_j{}^{\alpha_j}}{\beta_j{}^{2\alpha_j-1}} t_j{}^{\alpha_j-1} exp\left\{-\left[\frac{r_j t_j}{\beta_j}\right]^{\alpha_j}\right\}\left\{\frac{1}{\sqrt{1-\rho^2}}(\exp[\frac{-\rho}{2(1-\rho^2)}\{\rho(z_1^2+z_2^2)-2z_1z_2\}])\right\}, \quad t_j,\alpha_j,\beta_j,r_j>0, \tag{18}$$

where $\rho \in [-1,1]$ is a dependence parameter. Fig. 1 displays the curve and the contour of BWEXP model.

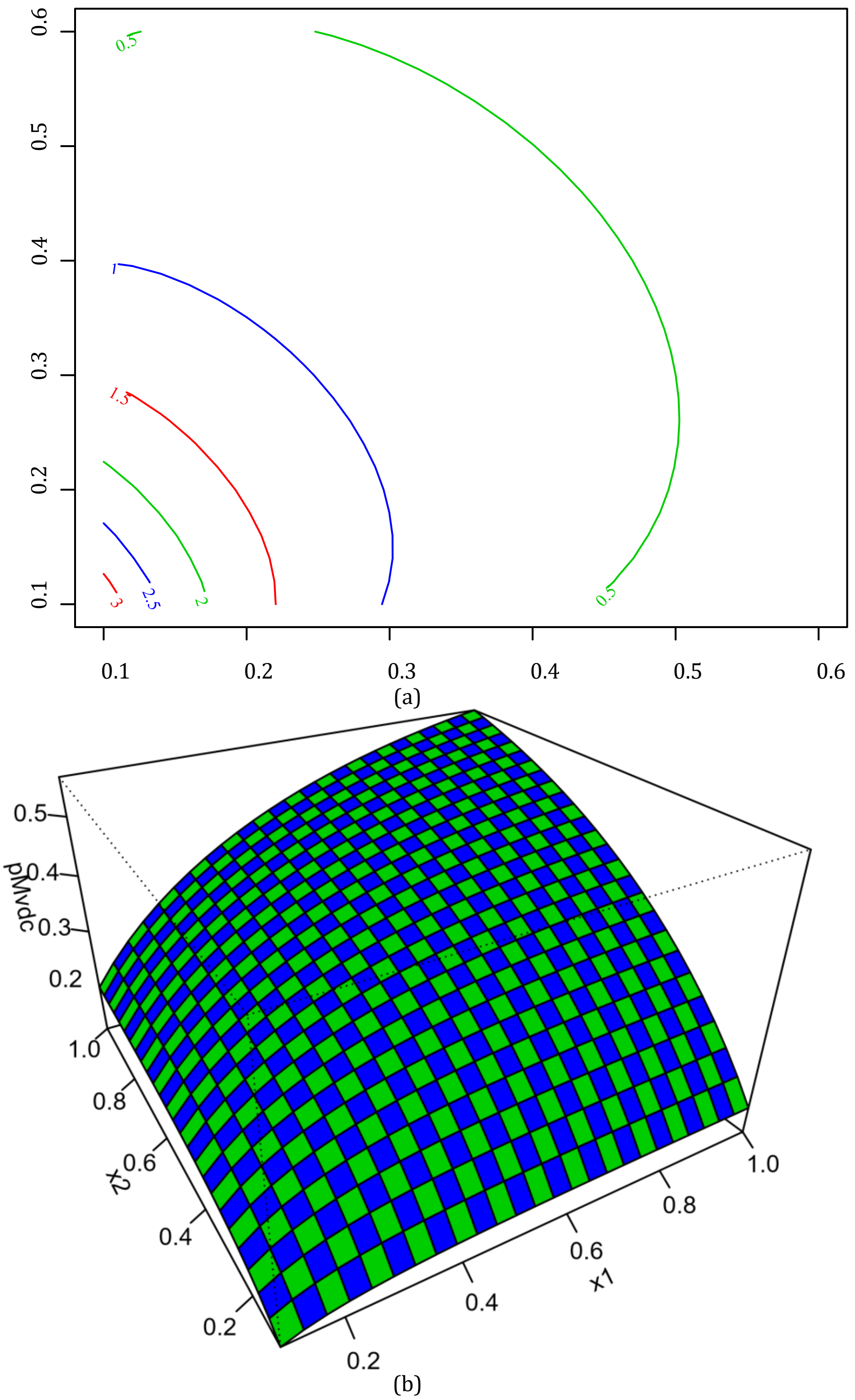

Fig. 1: (a) the contour plot of simulation data from BWE based on Gaussian copula (b) the PDF curve of simulation data from BWE based on Gaussian copula

4.3. Bivariate Weibull-Rayleigh distribution

Consider unimodal marginals of Weibull-Rayleigh (WR) distribution then the bivariate Weibull-Rayleigh (BWR) distribution PDF is

$$f\left(t_1,t_2,\underline{\alpha},\underline{\beta},\underline{r}\right) = \prod_{j=1}^{2}\frac{2\alpha_j r_j{}^{\alpha_j}}{\beta_j{}^{2\alpha_j-1}} t_j{}^{2\alpha_j-1} exp\left\{-\left[\frac{r_j t_j{}^2}{\beta_j}\right]^{\alpha_j}\right\}\left\{\frac{1}{\sqrt{1-\rho^2}}(\exp[\frac{-\rho}{2(1-\rho^2)}\{\rho(z_1^2+z_2^2)-2z_1z_2\}])\right\}, \quad t_j,\alpha_j,\beta_j,r_j>0, \tag{19}$$

where $\rho \in [-1,1]$ is a dependence parameter. Fig. 2 shows the curve and the contour of BWR model.

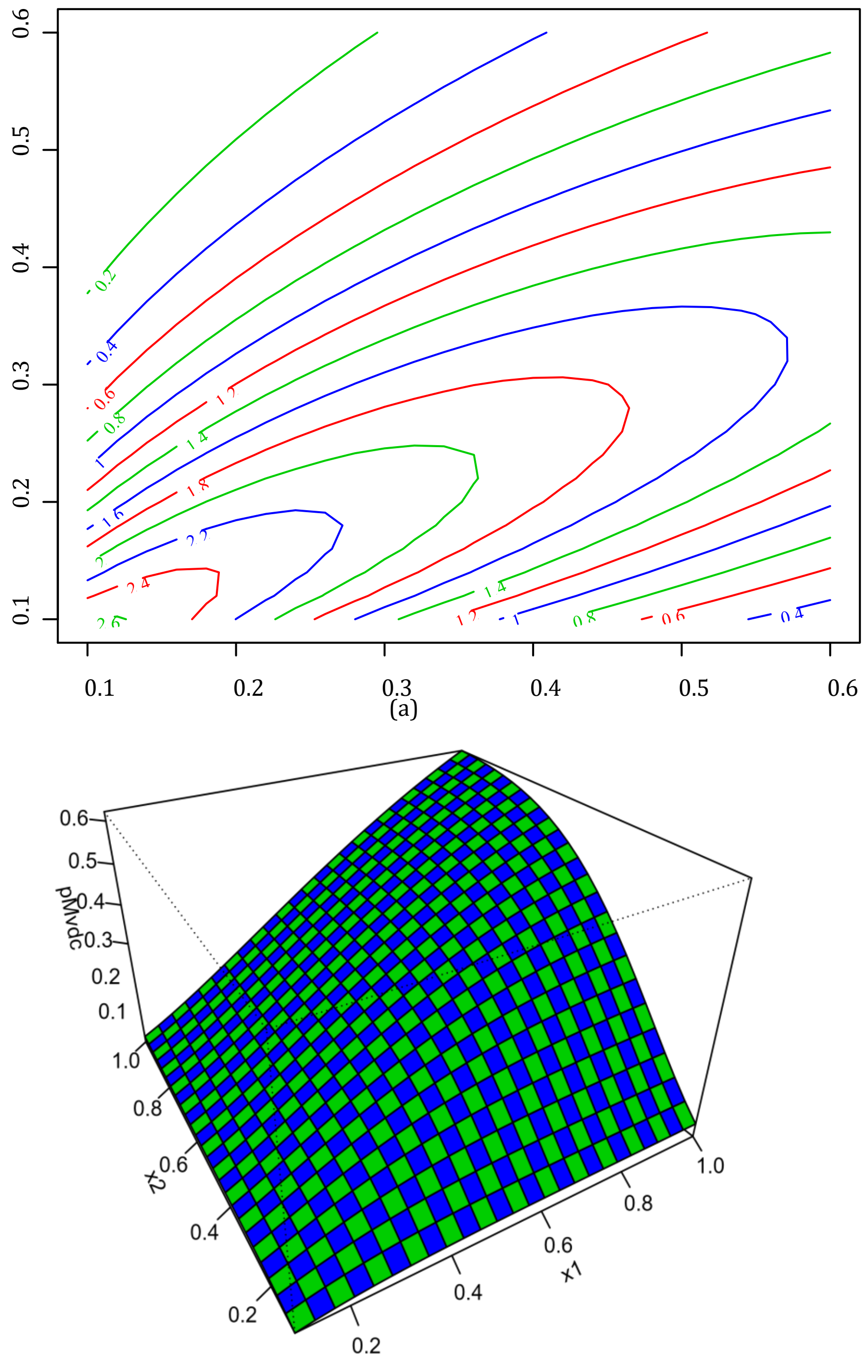

Fig. 2: (a) the contour plot of simulation data from BWR based on Gaussian copula (b) the PDF curve of simulation data from BWR based on Gaussian copula

4.4. Bivariate Weibull-Chi-square distribution

Assume that the marginals are Weibull- Chi-square (WCHI) distribution then the bivariate Weibull-Chi-square (BWCHI) distribution PDF is

$$f\left(t_1,t_2,\underline{\alpha},\underline{\beta},\underline{r}\right) = \prod_{j=1}^{2} \frac{\alpha_j \frac{2^{-\frac{r_j}{2}}}{\Gamma(\frac{r_j}{2})} t^{\frac{r_j}{2}-1} \exp\left(-\frac{t_j}{2}\right)}{\beta_j{}^{\alpha_j}\left[1-\frac{1-\Gamma(t_j,\frac{r_j}{2})}{\Gamma(\frac{r_j}{2})}\right]} \left\{-\frac{log\left[1-\frac{1-\Gamma(t_j,\frac{r_j}{2})}{\Gamma(\frac{r_j}{2})}\right]}{\beta_j}\right\}^{\alpha_j-1} exp\left\{-\left[-\frac{log\left[1-\frac{1-\Gamma(t_j,\frac{r_j}{2})}{\Gamma(\frac{r_j}{2})}\right]}{\beta_j}\right]^{\alpha_j}\right\} \left\{\frac{1}{\sqrt{1-\rho^2}}(\exp[\frac{-\rho}{2(1-\rho^2)}\{\rho(z_1^2+z_2^2)-2z_1\ z_2\ \}])\right\}, \quad (20)$$

where $t_j, \alpha_j, \beta_j, r_j > 0$.

Fig. 3 presents the curve and the contour of BWR model.

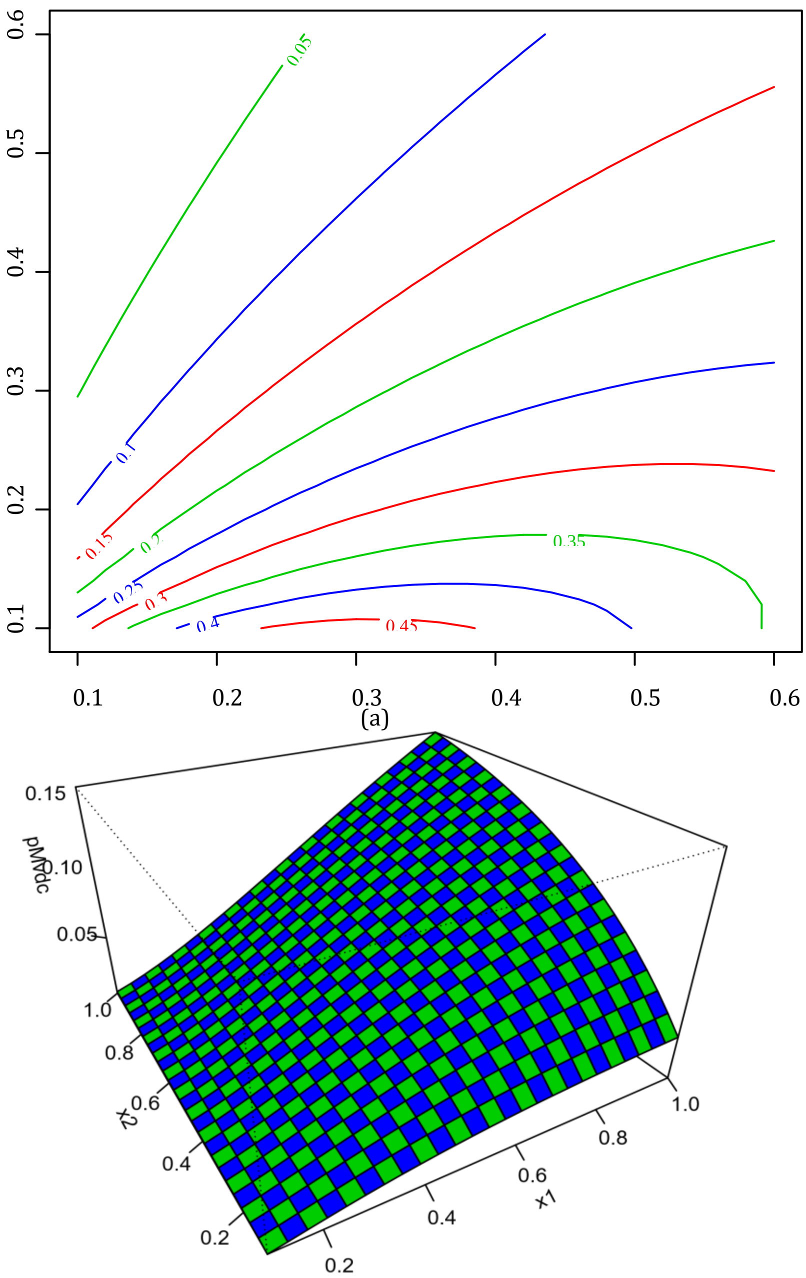

Fig. 3: (a) the contour plot of simulation data from BWCHI based on Gaussian copula (b) the PDF curve of simulation data from BWCHI based on Gaussian copula

5. Parameter estimation

In this section, we provide the estimation of the unknown parameters Bivariate Weibull-G distributions with Gaussian copula. There are two approaches to fit copula models. Parametric and semi-parametric are methods used to estimate proposed distribution parameters.

5.1. Parametric methods of estimation

We use two methods to fit copula models. The first method estimates the marginal parameters and the copula parameter through two steps, separately. The second approach has two steps to obtain the estimation of parameters of marginal and copula, where the estimation is computed from the pseudo-observations separately.

5.1.1. Maximum likelihood estimation (ML)

We provide the estimation of the unknown parameters of BWG distributions by the approach maximum likelihood estimation and use the two-step estimation (ML). This approach consists of two steps procedure, where the estimations of parameters of marginals are obtained separately from copula parameter.

The log-likelihood function expressed as

$$\log L = \sum_{i=1}^{n}[\log f_1(x_{1i}) + \log f_2(x_{2i}) + \log c(F_1(x_{1i}), F_2(x_{2i}))] \quad (21)$$

hence

$$log\, L = \sum_{i=1}^{n} log\, f_1(x_{1i}) + \sum_{i=1}^{n} log\, f_2(x_{2i}) + \sum_{i=1}^{n} log\, c(F_1(x_{1i}), F_2(x_{2i}))] \quad (22)$$

Firstly, we estimate the parameters of marginals via MLE approach separately, as given,

$$\log\, L_j = \sum_{i=1}^{n} \log f_j(x_{ji}), j = 1,2. \quad (23)$$

Secondly, copula parameters are estimated by differentiating the given L

$$\log L = \sum_{i=1}^{n} log\, c(F_1(x_{1i}), F_2(x_{2i})) \quad (24)$$

By considering the first step with (WG) family of distributions, the estimation of marginal parameters are computed by MLE method. Suppose $t_1, \ldots, t_n$ is a random sample fromWG(α_j,β_j,δ_j), then the log-likelihood function$L(\alpha_j,\beta_j, \delta_j)$ is

$$\log \quad L_j(t_j, \alpha_j, \beta_j, \delta_j) = nlog(\alpha_j) + n(2\alpha_j - 1)log(\beta_j) + \sum_{i=1}^{n} \log(g(t_{ji}, \delta_j)) - \sum_{i=1}^{n} \log(1 - G(t_j, \delta_j)) + (\alpha_j - 1) \sum_{i=1}^{n} \log\{-log[1 - G(t_{ji}, \delta_j)]\} - \sum_{i=1}^{n} \left[-\frac{log[1-G(t_{ji},\delta_j)]}{\beta_j}\right]^{\alpha_j}. \quad (25)$$

$$\frac{\partial \log L_j(t_j,\alpha_j,\beta_j,\delta_j)}{\partial \alpha_j} = \frac{n}{\alpha_j} + \sum_{i=1}^{n} \log\{-log[1 - G(t_{ji}, \delta_j)]\} + 2nlog(\beta_j) = 0\,. \quad (26)$$

$$\frac{\partial \log L_j(t_j,\alpha_j,\beta_j,\delta_j)}{\partial \beta_j} = \frac{n(2\alpha_j-1)}{\beta_j} + \frac{\alpha_j}{\beta_j{}^2} \sum_{i=1}^{n} \left[-\frac{log[1-G(t_{ji},\delta_j)]}{\beta_j}\right]^{\alpha_j-1} = 0. \quad (27)$$

$$\frac{\partial \log L_j(t_j, \alpha_j, \beta_j, \delta_j)}{\partial \delta_j} = \sum_{i=1}^{n} \left[\frac{\left(\frac{\partial\left(g(t_{ji}, \delta_j)\right)}{\partial \delta_{jk}}\right)}{g(t_{ji}, \delta_j)}\right] - \sum_{i=1}^{n} \left[\frac{\left(\frac{\partial\left(1 - G(t_{ji}, \delta_j)\right)}{\partial \delta_{jk}}\right)}{(1 - G(t_j, \delta_j)}\right] -(\alpha_j - 1) \sum_{i=1}^{n} \left[\frac{\left(\frac{\partial\left(1 - G(t_{ji}, \delta_j)\right)}{\partial \delta_{jk}}\right)}{\{-\log(1 - G(t_j, \delta_j)\}(1 - G(t_j, \delta_j)}\right] + \frac{\alpha_j}{\beta_j} \left[\frac{\left\{\left[-\frac{log[1-G(t_{ji},\delta_j)]}{\beta_j}\right]^{\alpha_j-1}\right\}\left(\frac{\partial\left(1-G(t_{ji},\delta_j)\right)}{\partial \delta_{jk}}\right)}{(1-G(t_j,\delta_j)}\right] = 0 \quad (28)$$

The solution of nonlinear equations (25), (26) and (27) gives the estimations of α_j, β_j and δ_j via MLE. Therefore,

$$\log L(\theta) = \sum_{i=1}^{n} log\, c\left(\widehat{F_1}(t_{1i}), \widehat{F}_2(t_{2i})\right) \quad (29)$$

where $\widehat{F_1}(t_1)$ and ,$\widehat{F}_2(t_2)$ denote as the ML of parameters estimations, note that these estimations should be obtained firstly. By equating the nonlinear Eq. 29 to zero, then the estimate of θ will be obtained using MLE method.

5.1.2 Modified maximum likelihood estimation (MML)

This method is suggested as following. In the first step, we use MLE approach to obtain marginal parameters, separately as given,

$$\log L_j = \sum_{i=1}^{n} \log f_j(x_{ji}), j = 1,2.$$

The parameter estimation via MLE is obtained by taking equations (26), (27) and (28) equal to zero. Since the formula is complicated, one can solve these equations numerically for α_j, β_j, and δ_j. Usually the statistician use Newton-Raphson algorithm as an iterative method obtain the estimates of these parameters as a first step.

Secondly, the estimations of copula parameters are obtained through the maximization of copula density as

$$\log L\,(\theta) = \sum_{i=1}^{n} \log[c_\theta(\widehat{U}_i, \widehat{V}_i)] \quad (30)$$

where $\widehat{H}_i$, $\widehat{\Psi}_i$ are pseudo-observations computed from

$$\widehat{M}_i = \frac{S_{1i}}{n+1} = \frac{n}{n+1} \widehat{F}_1\,(t_{1i}),$$
$$\widehat{K}_i = \frac{S_{2i}}{n+1} = \frac{n}{n+1} \widehat{F}_1\,(t_{2i}),$$

S_{1i} ,S_{2i} are known as the ranks of t_{1i}, t_{2i}, respectively. It is important, firstly obtain the margins CDF parametrically.

5.2. Semi-parametric methods of estimation

The semi-parametric methods estimate the copula parameter in copula models; Methods-of-moments approaches of namely inversion Kendall's and inversion of Spearman's rho.

5.2.1. Methods of moments

Method of moments is an approach of inversion Kendall's and inversion of Spearman's rho. With regard to Kojadinovic and Yan (2010). Let c be a bivariate random sample from CDF $C_\theta\,[F_1(t_1), F_2(t_2)]$, where F_1 and F_2 are continuous and continuous copula C_θ such that $\theta \in \mathcal{O}$, note that the vector $\mathcal{O}$ lie on an open interval in R^2. Furthermore, let $R_1, \ldots, R_n$ are ranks vector correspond to $t_1, \ldots, t_n$ unless other assumptions. In what follows, all vectors are known as row vectors. Method-of-moments approaches are known as a consistent estimator for the copula C_θ moment. The two best-known moments, Spearman's rho and Kendall's tau, are respectively given by

$$\rho(\theta) = 12 \int_{[0,1]^2} u \; v \, C_\theta(u,v) - 3 \,, \tag{31}$$

and

$$\tau(\theta) = 4 \int_{[0,1]^2} C_\theta(u,v) \, d \, C_\theta(u,v) - 1 \tag{32}$$

Now, the consistent estimators are computed as

$$\rho_n = \frac{12}{n(n+1)(n-1)} \sum_{i=1}^{n} R_{i,1} R_{i,2} - 3\frac{n+1}{n-1}, \tag{33}$$

and

$$\tau_n = \frac{4}{n(n-1)} \sum_{i=1}^{n} 1[t_{i,1} \le t_{j,1}] 1[t_{i,2} \le t_{j,2}] - 1, \tag{34}$$

When the functions ρ and τ are one-to-one, consistent estimators of θ are given by

$$\theta_{n,\rho} \; = \; \rho^{-1}(\rho_n), \quad \theta_{n,\tau} = \tau^{-1}(\tau_n).$$

It can be called inversion of Kendall's (itau) and inversion of Spearman's rho (irho) respectively. For more information, see Kojadinovic and Yan (2010). As explained above the methods-of-moments (itau) and (irho) estimation method for copula is considered as a semi-parametric method of estimation.

6. Goodness of fit tests for copula

The idea of this test is to compare the result of experimental for copula with the estimator obtained parametrically derived under the null hypothesis (Fermanian, 2005; Dobrić and Schmid, 2007). Now, analyze if C provides well-represented for a certain copula C_θ

$$H_0 \colon C = C_\theta \qquad Vs. \;\; H_1 \colon C \neq C_\theta$$

Two approaches are commonly used in the literature to test the goodness of fit of a copula; the parametric bootstrap (Genest and Rémillard, 2008) or the fast multiplier approach (Kojadinovic et al., 2011; Genest et al., 2009). The goodness of fit tests based on the empirical process

$$\mathbb{C}_n(m,k) = \sqrt{n}\{C_n(m,k) - C_{\theta_n}(m,k)\},$$

where $C_n(m,k)$ is known as empirical copula of T_1 and T_2

$$C_n(m,k) = \frac{1}{n}\sum_{i=1}^{n} 1\big(M_{i,n} \le m, K_{i,n} \le k\big), \quad m,k \in [0,1],$$

$M_{i,n}, M_{i,n}$are pseudo-observations computed from C as follows

$$M_{i,n} = \frac{S_{1i}}{n+1},$$

$$K_{i,n} = \frac{S_{2i}}{n+1}$$

where S_{1i} ,S_{2i} are respectively the ranks of t_{1i}, t_{2i}.

The θ_n is defined as estimates of θ through pseudo-observations, and $C_n(m,k)$ provides a consistent estimator. According to Genest et al. (2009), $U_n = \sum_{i=1}^{n}\{C_n(M_{i,n}, K_{i,n}) - C_{\theta_n}(M_{i,n}, K_{i,n})\}^2$ is the test statistics. For details see Genest, et al. (1995, 2009) and Kojadinovic et al. (2011).

7. Simulation study

The simulation study discusses the comparison of proposed models of WG one parameter marginal distributions based on Gaussian copulas. The correlation coefficients of Spearman's rho measure and also Kendall's tau measure are considered. Both measures are computed to obtain the values of copula parameters.

Considering the following values of marginal and copula parameters of BWG distributions with different sizes of sample (n = 30, 50, 100, and 150) with Gaussian copula parameter $\theta_G = 0.8$. In this implementation, we estimate the parameters of three proposed models. In addition, we compute the properties of estimations; variances, the property mean squared errors (MSE) of estimations, bias of estimations, and relative mean squared errors of parameters estimations (RMSE), this case is replicated 1000 times. The results of simulation are displayed in Tables 1 to 7.

8. Results and discussion

From simulation results we observe the following:

1. As expected, most results improve with increases in sample size.
2. For most selected values of $\alpha_1, b_1,\ r_1, \alpha_2, b_2, r_2$ and θ_G, the estimation properties involve RMSE, MSE and bias of the estimates $\hat{\alpha}_1, \hat{b}_1, \hat{r}_1, \hat{\alpha}_2, \hat{b}_2, \hat{r}_2$ and $\hat{\theta}_G$ become smaller as the sample size increases.
3. Table 1 shows, for α_1 greater than α_2, the most results $\widehat{\alpha_1}$ are generally better than $\widehat{\alpha_2}$, and the values of $\widehat{\alpha_1}$ rapidly get better than $\widehat{\alpha_2}$ when n is increased. The same conclusions we can observe for b_2 which is greater than b_1, also the result of $\widehat{r_1}$ is better than $\widehat{r_2}$. The estimate of θ_G is presented in Table 2.
4. Table 3 illustrates that α_2 is greater thanα_1, the results of $\widehat{\alpha_1}$ are better than $\widehat{\alpha_2}$. When n is increased the values of $\widehat{\alpha_1}$ get better more rapidly. As well as results of $\hat{b}_1$ are generally better than $\hat{b}_2$. The estimates of r_1 greater than r_2, and the most results $\widehat{r_1}$ are suitable. The estimate of θ_G is presented in Table 4.

Table 1: The estimates, bias, MSE and RMSE of parameters for BWE distribution

Sample Size		Estimates, bias, mean square errors and relative mean square errors of Parameters						
		$\alpha_1 = 0.8$	$b_1 = 0.7$	$r_1 = 0.9$	$\alpha_2 = 0.7$	$b_2 = 0.8$	$r_2 = 0.75$	$\theta_G = 0.8$
n=30	ML	0.734	0.837	0.932	0.839	0.704	0.764	0.597
		0.066	0.137	0.032	0.139	0.096	0.062	0.203
		0.016	0.108	0.044	0.035	0.053	0.062	0.041
		0.021	0.155	0.048	0.050	0.067	0.083	0.052
	MML	0.734	0.837	0.932	0.839	0.704	0.764	0.813
		0.066	0.137	0.032	0.139	0.096	0.062	0.013
		0.016	0.108	0.044	0.035	0.053	0.062	0.000
		0.021	0.155	0.048	0.050	0.067	0.083	0.000
n=50	ML	0.720	0.954	1.061	0.820	0.728	0.781	0.525
		0.080	0.254	0.161	0.120	0.072	0.006	0.275
		0.013	0.238	0.160	0.024	0.020	0.006	0.076
		0.017	0.340	0.177	0.034	0.025	0.008	0.095
	MML	0.720	0.954	0.061	0.820	0.728	0.781	0.771
		0.080	0.254	0.161	0.120	0.072	0.006	0.029
		0.013	0.238	0.160	0.024	0.020	0.006	0.000
		0.017	0.340	0.177	0.034	0.025	0.008	0.001
n=100	ML	0.711	0.683	0.786	0.811	0.760	0.822	0.812
		0.089	0.017	0.114	0.111	0.040	0.094	0.012
		0.011	0.014	0.057	0.017	0.057	0.094	0.000
		0.014	0.020	0.064	0.024	0.072	0.126	0.000
	MML	0.711	0.683	0.786	0.811	0.760	0.822	0.781
		0.089	0.017	0.114	0.111	0.040	0.094	0.019
		0.011	0.014	0.057	0.017	0.057	0.094	0.000
		0.014	0.020	0.064	0.024	0.072	0.126	0.000
n=150	ML	0.706	0.864	0.970	0.806	0.620	0.660	0.628
		0.094	0.164	0.070	0.106	0.180	0.025	0.172
		0.148	0.750	0.441	0.189	0.707	0.334	0.398
		0.185	1.071	0.490	0.270	0.886	0.445	0.498
	MML	0.706	0.864	0.970	0.806	0.620	0.660	0.804
		0.094	0.164	0.070	0.106	0.180	0.025	0.004
		0.148	0.750	0.441	0.189	0.707	0.334	0.008
		0.185	1.071	0.490	0.270	0.886	0.445	0.011

Table 2: The estimation study of correlation parameter for BWE distribution

Sample Size	$\theta_G = 0.8$				
	Estimates	bias	MSE	RMSE	Estimation Methods
n=30	0.597	0.203	0.041	0.052	ML
	0.813	0.013	0.000	0.000	MML
	0.817	0.017	0.000	0.000	Itau
	0.821	0.021	0.000	0.001	IRho
n=50	0.525	0.275	0.076	0.095	ML
	0.771	0.029	0.001	0.001	MML
	0.755	0.045	0.002	0.003	Itau
	0.762	0.038	0.001	0.002	IRho
n=100	0.812	0.012	0.000	0.000	ML
	0.781	0.019	0.000	0.000	MML
	0.777	0.023	0.001	0.001	Itau
	0.776	0.024	0.001	0.001	IRho
n=150	0.628	0.172	0.398	0.498	ML
	0.804	0.004	0.008	0.011	MML
	0.806	0.006	0.021	0.026	Itau
	0.804	0.004	0.012	0.015	IRho

Table 3: The estimates, bias, MSE and RMSE of parameters for BWR distribution

Sample Size		Estimates ,bias, mean square errors and relative mean square errors of Parameters						
		$\alpha_1 = 0.7$	$b_1 = 0.8$	$r_1 = 0.8$	$\alpha_2 = 0.8$	$b_2 = 0.7$	$r_2 = 0.5$	$\theta_G = 0.8$
n=30	ML	0.868	0.734	0.835	0.324	0.839	0.936	0.359
		0.168	0.066	0.035	0.476	0.139	0.207	0.441
		0.148	0.016	0.039	0.231	0.035	0.207	0.195
		0.212	0.021	0.049	0.289	0.050	0.413	0.243
	MML	0.868	0.734	0.835	0.324	0.839	0.936	0.813
		0.168	0.066	0.035	0.476	0.139	0.207	0.013
		0.148	0.016	0.039	0.231	0.035	0.207	0.000
		0.212	0.021	0.049	0.289	0.050	0.413	0.000
n=50	ML	0.951	0.720	0.927	0.471	0.820	1.384	0.417
		0.251	0.080	0.127	0.329	0.120	0.858	0.383
		0.238	0.013	0.135	0.109	0.024	0.858	0.147
		0.340	0.017	0.169	0.137	0.034	1.717	0.184
	MML	0.951	0.720	0.927	0.471	0.820	1.384	0.771
		0.251	0.080	0.127	0.329	0.120	0.858	0.029
		0.238	0.013	0.135	0.109	0.024	0.858	0.001
		0.340	0.017	0.169	0.137	0.034	1.717	0.001
n=100	ML	0.685	0.711	0.684	0.631	0.811	1.745	0.069
		0.015	0.089	0.116	0.169	0.111	2.263	0.731

		0.017	0.011	0.052	0.134	0.017	2.263	0.535
		0.024	0.014	0.065	0.168	0.024	4.525	0.668
	MML	0.685	0.711	0.684	0.631	0.811	1.745	0.781
		0.015	0.089	0.116	0.169	0.111	2.263	0.019
		0.017	0.011	0.052	0.134	0.017	2.263	0.000
		0.024	0.014	0.065	0.168	0.024	4.525	0.000
n=150	ML	0.893	0.706	0.870	0.321	0.806	0.919	0.455
		0.193	0.094	0.070	0.479	0.106	0.191	0.345
		0.908	0.148	0.359	0.850	0.189	0.701	0.438
		1.297	0.185	0.449	1.063	0.270	1.402	0.547
	MML	0.893	0.706	0.870	0.321	0.806	0.919	0.804
		0.193	0.094	0.070	0.479	0.106	0.191	0.004
		0.908	0.148	0.359	0.850	0.189	0.701	0.008
		1.297	0.185	0.449	1.063	0.270	1.402	0.011

5. Now, Table 5 displays α_2 greater than α_1, the most results $\widehat{\alpha_2}$ are generally better than $\widehat{\alpha_1}$, and the values of $\widehat{\alpha_2}$ get better more rapidly than $\widehat{\alpha_1}$ as the sample size increases. For b_1 greater than b_2, the results $\hat{b}_1$ look better than $\hat{b}_2$, the values of $\hat{b}_1$ get better more rapidly than the values of $\hat{b}_2$ as the sample size increases. Similarly, for r_1 greater than r_2, the most results $\widehat{r_2}$ for are generally better than $\widehat{r_1}$ for Furthermore, the values of $\widehat{r_2}$ get better more rapidly than the values of $\widehat{r_1}$. The estimate of correlation model is displayed in Table 6.

Table 4: The estimation study of correlation parameter for BWE distribution merged with copula

Sample Size	$\theta_G = 0.8$				
	Estimates	bias	*MSE*	RMSE	Estimation Methods
n=30	0.359	0.441	0.195	0.243	ML
	0.813	0.013	0.000	0.000	MML
	0.817	0.017	0.000	0.000	Itau
	0.821	0.021	0.000	0.001	IRho
n=50	0.417	0.383	0.147	0.184	ML
	0.771	0.029	0.001	0.001	MML
	0.755	0.045	0.002	0.003	Itau
	0.762	0.038	0.001	0.002	IRho
n=100	0.069	0.731	0.535	0.668	ML
	0.781	0.019	0.000	0.000	MML
	0.777	0.023	0.001	0.001	Itau
	0.776	0.024	0.001	0.001	IRho
n=150	0.455	0.345	0.438	0.547	ML
	0.804	0.004	0.008	0.011	MML
	0.806	0.006	0.021	0.026	Itau
	0.804	0.004	0.012	0.015	IRho

Table 5: The results of simulation study for BWCHI distribution

Sample Size		Estimates, bias, mean square errors and relative mean square errors of Parameter						
		$\alpha_1 = 0.7$	$b_1 = 0.8$	$r_1 = 4$	$\alpha_2 = 0.8$	$b_2 = 0.7$	$r_2 = 3$	$\theta_G = 0.8$
n= 30	ML	1.300	0.899	4.249	1.158	1.052	3.204	0.511
		0.600	0.099	0.249	0.358	0.352	5.532	0.289
		1.904	1.159	8.333	1.297	2.060	5.532	1.134
		2.719	1.449	2.083	1.622	2.943	1.844	0.385
	MML	1.300	0.899	4.249	1.158	1.052	3.204	0.763
		0.600	0.099	0.249	0.358	0.352	5.532	0.037
		1.904	1.159	8.333	1.297	2.060	5.532	0.067
		2.719	1.449	2.083	1.622	2.943	1.844	0.083
n= 50	ML	1.22	0.811	4.300	1.004	0.943	3.200	0.547
		0.422	0.0107	0.300	0.204	0.243	4.176	0.253
		1.107	0.150	6.659	0.725	0.254	4.176	0.064
		1.582	0.188	1.665	0.906	0.363	1.392	0.080
	MML	1.22	0.811	4.300	1.004	0.943	3.200	0.771
		0.422	0.0107	0.300	0.204	0.243	4.176	0.029
		1.107	0.150	6.659	0.725	0.254	4.176	0.001
		1.582	0.188	1.665	0.906	0.363	1.392	0.001
n= 100	ML	0.964	0.747	4.274	0.866	0.870	3.194	0.669
		0.264	0.053	0.274	0.068	0.170	2.831	0.131
		0.521	0.067	4.412	0.370	0.127	2.831	0.017
		0.745	0.084	1.103	0.463	0.182	0.944	0.021
	MML	0.964	0.747	4.274	0.866	0.870	3.194	0.781
		0.264	0.053	0.274	0.068	0.170	2.831	0.019
		0.521	0.067	4.412	0.370	0.127	2.831	0.000
		0.745	0.084	1.103	0.463	0.182	0.944	0.000
n= 150	ML	0.926	0.735	4.155	0.807	0.841	3.166	0.670
		0.226	0.065	0.155	0.007	0.141	2.0253	0.130
		0.447	0.622	2.912	0.884	1.108	2.025	0.229
		1.852	0.777	2.679	1.105	0.430	2.484	0.286
	MML	0.926	0.735	4.155	0.807	0.841	3.166	0.804

	0.226	0.065	0.155	0.007	0.141	2.0253	0.004
	0.447	0.622	2.912	0.884	1.108	2.025	0.008
	1.852	0.777	2.679	1.105	0.430	2.484	0.011

Table 6: The results of correlation parameter for BWCHI distribution

Sample Size	$\theta_G = 0.8$				
	Estimates	bias	MSE	RMSE	Methods of Estimation
n=30	0.511	0.289	1.134	0.385	ML
	0.763	0.037	0.067	0.083	MML
	0.796	0.004	0.010	0.013	Itau
	0.797	0.003	0.022	0.007	IRho
n=50	0.547	0.253	0.064	0.080	ML
	0.771	0.029	0.001	0.001	MML
	0.755	0.045	0.002	0.003	Itau
	0.762	0.038	0.001	0.002	IRho
n=100	0.669	0.131	0.017	0.021	ML
	0.781	0.019	0.000	0.000	MML
	0.777	0.023	0.001	0.001	Itau
	0.776	0.024	0.001	0.001	IRho
n=150	0.670	0.130	0.229	0.286	ML
	0.804	0.004	0.008	0.011	MML
	0.806	0.006	0.021	0.026	Itau
	0.804	0.004	0.012	0.015	IRho

6. We remark that the efficient estimators of marginal parameters of three models differ according to the parameters. It seems that ML estimates$\hat{\alpha}_1, \hat{b}_1, \hat{r}_1, \hat{\alpha}_2, \hat{b}_2, \hat{r}_2$ and of three models are the same corresponding MML estimates.
7. For copula parameter, the MML provides efficient most estimates than ML, Itau, and Irho, for all models with different marginals parameters and Gaussian copula. It is also note that the ML and MML estimates for all copula parameters are close.
8. For Gaussian, copula parameters it is observe that to Itau, and Irho estimates of three models are the same corresponding three models.

To check if the selected parametric copula function is suitable for the marginals, goodness of fit test statistics using selected copula function for the marginals is preformed. The results in Table 7 shows a non-significant p-value obtained using parametric bootstrap for all models which indicate that selected parametric copula function provide appropriate fit to the marginals. In addition, estimate of the copula parameter based on ML, MML, Itau, and Irho methods for the Gussian copula for all models is approximately equals. This estimates are used as initial value when fitting these copula models using BWG marginals.

Table 7: The results of goodness of fit for selected copula functions.

Model	statistic	p-value	Estimate of copula parameter θ	Estimation methods
BWE	0.0235	0.3272	0.7949	Ml
	0.0235	0.3422	0.79485	MML
	0.0270	0.2792	0.7548	Itau
	0.0261	0.3651	0.7625	Irho
BWR	0.0235	0.3272	0.7949	Ml
	0.0235	0.3422	0.7949	MML
	0.0270	0.2792	0.7548	Itau
	0.0261	0.3651	0.7625	Irho
BWCHI	0.0235	0.3272	0.7949	Ml
	0.0235	0.3422	0.7949	MML
	0.0270	0.2792	0.7548	Itau
	0.0261	0.3651	0.7625	Irho

9. Conclusion

This paper presents a family of distributions with one parameter which construct Bivariate Weibull-G distributions merging with Gaussian copula. The characteristics are discussed and the parameters estimations are obtained via two approaches, parametric and semi-parametric methods. To study the effectiveness of models, the simulation cases are presented with different values of model parameters and different sample sizes. One can conclude when the sample size is increased the results become more consistent. Moreover, the selection of copula function is suitable for the bivariate models. To sum up. The proposed models provide high flexibility and could deal with different structure type of data set. In future work, implementing other distribution with more than parameters to construct Bivariate Weibull-G models, also modifying the models for multivariate distribution.

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